

REMARKS ON NONLINEAR EQUATIONS WITH MEASURES

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To the memory of I. V. Skrypnik

ABSTRACT. We study the Dirichlet boundary value problem for equations with absorption of the form $-\Delta u + g \circ u = \mu$ in a bounded domain $\Omega \subset \mathbb{R}^N$ where g is a continuous odd monotone increasing function. Under some additional assumptions on g , we present necessary and sufficient conditions for existence when μ is a finite measure. We also discuss the notion of solution when the measure μ is positive and blows up on a compact subset of Ω .

1. INTRODUCTION

In this paper we discuss some aspects of the boundary value problem

$$(1.1) \quad \begin{aligned} -\Delta u + g \circ u &= \mu & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega, \end{aligned}$$

where $\mu \in \mathfrak{M}_\rho(\Omega)$, i.e. μ is a Borel measure such that

$$\int_{\Omega} \rho d|\mu| < \infty, \quad \rho(x) = \text{dist}(x, \partial\Omega).$$

In addition we define a notion of solution in the case that μ is a positive Borel measure which may explode on a compact subset of the domain and discuss the question of existence and uniqueness in this case. We always assume that $g \in C(\mathbb{R})$ is a monotone increasing function such that $g(0) = 0$. To simplify the presentation we also assume that g is odd.

A function $u \in L^1(\Omega)$ is a weak solution of the boundary value problem (1.1), $\mu \in \mathfrak{M}_\rho$, if $u \in L_\rho^g(\Omega)$, i.e.

$$\int_{\Omega} g(u) \rho dx < \infty$$

and

$$(1.2) \quad \int_{\Omega} (-v \Delta \phi + g \circ v \phi) dx = \int_{\Omega} \phi d\mu$$

for every $\phi \in C_0^2(\bar{\Omega})$ ($=$ space of functions in $C^2(\bar{\Omega})$ vanishing on $\partial\Omega$).

We say that u is a solution of the equation

$$(1.3) \quad -\Delta u + g \circ u = \mu \quad \text{in } \Omega$$

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if u and $g \circ u$ are in $L^1_{\text{loc}}(\Omega)$ and (1.2) holds for every $\phi \in C_c^2(\Omega)$.

Brezis and Strauss [6] proved that, if μ is an L^1 function the problem possesses a unique solution. This result does not extend to arbitrary measures in $\mathfrak{M}_\rho(\Omega)$.

Denote by \mathfrak{M}_ρ^g the set of measures $\mu \in \mathfrak{M}_\rho$ for which (1.1) is solvable. A measure in \mathfrak{M}_ρ^g is called a *g-good measure*. It is known that, if a solution exists then it is unique.

We say that g is *subcritical* if $\mathfrak{M}_\rho^g = \mathfrak{M}_\rho$. Benilan and Brezis, [5] and [4] proved that the following condition is sufficient for g to be subcritical:

$$(1.4) \quad \int_0^1 g(r^{2-N}) r^{N-1} dr < \infty.$$

In the case that g is a power non-linearity, i.e., $g = g_q$ where

$$g_q(t) = |t|^q \text{sign } t \quad \text{in } \mathbb{R}, \quad q > 1,$$

this condition means that $q < q_c := N/(N-2)$. Benilan and Brezis also proved that, if $g = g_q$ and $q \geq q_c$, problem (1.1) has no solution when μ is a Dirac measure.

Later Baras and Pierre [3] gave a complete characterization of \mathfrak{M}_ρ^g in the case that $g = g_q$ with $q \geq q_c$. They proved that a finite measure μ is g_q -good if and only if $|\mu|$ does not charge sets of $\bar{C}_{2,q'}$ capacity zero, $q' = q/(q-1)$. Here $\bar{C}_{\alpha,p}$ denotes Bessel capacity with the indicated indices.

In the present paper we extend the result of Baras and Pierre to a large class of non-linearities and also discuss the notion of solution in the case that μ is a positive measure which explodes on a compact subset of Ω .

2. STATEMENT OF RESULTS

Denote by \mathcal{H} the set of even functions h such that

$$(2.1) \quad \begin{aligned} h &\in C^1(\mathbb{R}), \quad h(0) = 0, \quad h \text{ is strictly convex,} \\ h'(0) &= 0, \quad h'(t) > 0 \quad \forall t > 0, \quad \lim_{t \rightarrow \infty} h'(t) = \infty. \end{aligned}$$

For $h \in \mathcal{H}$ denote by $L^h(\Omega)$ the corresponding Orlicz space in a domain $\Omega \subset \mathbb{R}^N$:

$$L^h(\Omega) = \{f \in L^1_{\text{loc}}(\Omega) \mid \exists k > 0 : h \circ (f/k) \leq 1\}$$

with the norm

$$\|f\|_{L^h} = \inf\{k > 0 \mid h \circ (f/k) < \infty\}.$$

Further denote by h^* the conjugate of h . Since, by assumption, h is strictly convex, h' is strictly increasing so that,

$$h^*(t) = \int_0^t (h')^{-1}(s) ds.$$

Let G be the Green kernel for $-\Delta$ in Ω and denote

$$\mathbb{G}_\mu(x) = \int_\Omega G(x, y) d\mu(y) \quad \forall x \in \Omega, \quad \mu \in \mathfrak{M}_\rho(\Omega).$$

For every $h \in \mathcal{H}$, the capacity $C_{2,h}$ in Ω is defined as follows. For every compact set $E \subset \Omega$ put:

$$(2.2) \quad C_{2,h}(E) = \sup\{\mu(\Omega) : \mu \in \mathfrak{M}(\Omega), \mu \geq 0, \mu(E^c) = 0, \|\mathbb{G}\mu\|_{L^{h^*}} \leq 1\}.$$

If O is an open set:

$$C_{2,h}(O) = \sup\{C_{2,h}(E) : E \subset O, E \text{ compact.}\}$$

For an arbitrary set $A \subset \Omega$ put

$$C_{2,h}(A) = \inf\{C_{2,h}(O) : A \subset O \subset \Omega, O \text{ open.}\}$$

This definition is compatible with (2.2) : when E is compact the value of $C_{2,h}(E)$ given by the above formula coincides with the value given by (2.2), (see [2]).

We say that h satisfies the Δ_2 condition if there exists $C > 0$ such that

$$h(a+b) \leq c(h(a) + h(b)) \quad \forall a, b > 0.$$

If $h \in \mathcal{H}$ satisfies this condition then, L^h is separable (see [8]) and the capacity $C_{2,h}$ has the following additional properties (see [2]).

Let Ω be a bounded domain in \mathbb{R}^N . For every $A \subset \Omega$,

$$(2.3) \quad C_{2,h}(A) = \sup\{C_{2,h}(E) : E \subset A, E \text{ compact}\}$$

and for every increasing sequence of sets $\{A_n\}$

$$(2.4) \quad \lim C_{2,h}(A_n) = C_{2,h}(\cup A_n).$$

Furthermore, for every $A \subset \Omega$

$$(2.5) \quad C_{2,h}(A) = \inf\{\|f\|_{L^h} : f \in L^h(\Omega), \mathbb{G}f \geq 1 \text{ on } A\}.$$

If $h \in \mathcal{H}$ and both h and h^* satisfy the Δ_2 condition then L^h is reflexive [8].

Finally we denote by \mathcal{G} the space of odd functions in $C(\mathbb{R})$ such that $h := |g| \in \mathcal{H}$ and by \mathcal{G}_2 the set of functions $g \in \mathcal{G}$ such that h and h^* satisfy the Δ_2 condition. For $g \in \mathcal{G}$ put

$$L^g := L^{|g|}, \quad C_{2,g} := C_{2,|h|}, \quad g^*(t) = |g|^*(t) \text{sign } t \quad \forall t \in \mathbb{R}.$$

In the sequel we assume that Ω is a bounded domain of class C^2 . The first theorem provides a necessary and sufficient condition for the existence of a solution of (1.1) in the spirit of [3].

Theorem 2.1. *Let $g \in \mathcal{G}_2$ and let μ be a measure in $\mathfrak{M}_\rho(\Omega)$. Then problem (1.1) possesses a solution if and only if μ vanishes on every compact set $E \subset \Omega$ such that $C_{2,g^*}(E) = 0$. This condition will be indicated by the notation $\mu \prec C_{2,g^*}$.*

Next we consider problem (1.1) when μ is a positive Borel measure which may explode on a compact set $F \subset \Omega$. In this part of the paper we assume

that $g \in \mathcal{G}_2$ and that g satisfies the Keller – Osserman condition [9] and [12]. This condition ensures that the set of solutions of

$$(2.6) \quad -\Delta u + g \circ u = 0$$

in Ω is uniformly bounded in compact subsets of Ω . Therefore, if $E \subset \Omega$ and E is compact then there exists a maximal solution of

$$(2.7) \quad -\Delta u + g \circ u = 0 \quad \text{in } \Omega \setminus E, \quad u = 0 \quad \text{on } \partial\Omega.$$

This solution will be denoted by U_E .

Notation. Consider the family of positive Borel measures μ in Ω such that:

- (1) There exists a compact set $F \subset \Omega$ such that, for every open set $O \supset F$, $\mu(\Omega \setminus \bar{O}) < \infty$ and
- (2) $\mu(A) = \infty$ for every non-empty Borel set $A \subset F$.

The set F will be called the singular set of μ . The family of measures μ of this type will be denoted by $\mathcal{B}_\infty(\Omega)$.

Definition 2.2. Assume that $g \in \mathcal{G}$ and that g satisfies the Keller – Osserman condition. If $\nu \in \mathfrak{M}_\rho^g(\Omega)$ denote by v_ν the solution of (1.1) with μ replaced by ν .

Let $\mu \in \mathcal{B}_\infty(\Omega)$ and let F be the singular set of μ . A function $u \in L_{\text{loc}}^1(\bar{\Omega} \setminus F)$ (i.e., $u \in L^1(\Omega \setminus \bar{O})$ for every neighborhood O of F) is a generalized solution of (1.1) if:

- (i) u satisfies (1.2) for every $\phi \in C_0^2(\bar{\Omega})$ such that $\text{supp } \phi \subset \Omega \setminus F$.
- (ii) $u \geq V_F := \sup\{v_\nu : \nu \in \mathfrak{M}_\rho^g(\Omega), \nu \geq 0, \text{supp } \nu \subset F\}$.

Theorem 2.3. Assume that $g \in \mathcal{G}_2$ and that g satisfies the Keller – Osserman condition. Let $\mu \in \mathcal{B}_\infty$ with singular set F . Then:

- (i) Problem (1.1) has a generalized solution if and only if μ vanishes on every compact set $E \subset \Omega \setminus F$ such that $C_{2,g^*}(E) = 0$.

If $V_F = U_F$, where V_F is defined as in Definition 2.2 and U_F is the maximal solution associated with F (see (2.7)) then the generalized solution is unique.

- (ii) If g satisfies the subcriticality condition (1.4) then problem (1.1) possesses a unique generalized solution for every $\mu \in \mathcal{B}_\infty$.

- (iii) Let $g = g_q$, $q \geq q_c$. If $\mu \prec C_{2,g^*}$ in $\Omega \setminus F$ then (1.1) possesses a unique solution.

3. PROOF OF THEOREM 2.1

The proof is based on several lemmas. We assume throughout that the conditions of the theorem are satisfied.

Denote by $L_\rho^1(\Omega)$ the Lebesgue space with weight ρ and by $L_\rho^g(\Omega)$ the Orlicz space with weight ρ .

Further denote by $W^k L^g(\Omega)$, $k \in \mathbb{N}$, the Orlicz-Sobolev space consisting of functions $v \in L^g(\Omega)$ such that $D^\alpha v \in L^g(\Omega)$ for $|\alpha| \leq k$.

Under our assumptions the set of bounded functions in L^g is dense in this space (see [8]). Consequently, by [7], $C^\infty(\bar{\Omega})$ is dense in $W^k L^g(\Omega)$. As

a consequence of the reflexivity of L^g the space $W^k L^g(\Omega)$ is reflexive. Let $W_0^k L^g(\Omega)$ denote the closure of $C_c^\infty(\Omega)$ in $W^k L^g(\Omega)$. The dual of this space, denoted by $W^{-k} L^{g*}(\Omega)$ is the linear hull of $\{D^\alpha f : f \in L^{g*}(\Omega), |\alpha| \leq k\}$. The standard norm in $W^k L^g(\Omega)$ is given by

$$\|v\|_{W^k L^g} = \sum_{|\alpha| \leq k} \|D^\alpha v\|_{L^g}$$

and the norm in $W^{-k} L^{g*}$ is defined as the norm of the dual space of $W_0^k L^g$.

The spaces $W^k L_\rho^g$ and $W^{-k} L_\rho^{g*}$ are defined in the same way.

Lemma 3.1. *If $\mu \in \mathfrak{M}_\rho(\Omega)$ is a g -good measure then (1.1) has a unique solution, which we denote by v_μ . The solution satisfies the inequality*

$$(3.1) \quad \|v_\mu\|_{L^1(\Omega)} + \|v_\mu\|_{L_\rho^g(\Omega)} \leq C \|\mu\|_{\mathfrak{M}_\rho(\Omega)}$$

where C is a constant depending only on g and Ω .

If $\mu_j \in \mathfrak{M}_\rho(\Omega)$, $j = 1, 2$ are g -good measures and $\mu_1 \leq \mu_2$ then $v_{\mu_1} \leq v_{\mu_2}$.

These results are well-known (see e.g. [13]).

Lemma 3.2. *Let $\mu \in \mathfrak{M}_\rho(\Omega)$ be a positive measure such that $\mathbb{G}_\mu \in L_{\text{loc}}^g(\Omega)$. Then μ is g good.*

Proof. Let $\{\Omega_n\}$ be a C^2 uniform exhaustion of Ω . Then $\mathbb{G}_\mu \in L^g(\Omega_n)$ is a positive supersolution of problem (1.1) in Ω_n . Therefore – as the zero function is a subsolution – there exists a solution, say u_n , of (1.1) in Ω_n and, by Lemma 3.1,

$$\int_{\Omega_n} u_n dx + \int_{\Omega_n} g \circ u_n \rho_n dx \leq C \int_{\Omega_n} \rho_n d\mu,$$

where $\rho_n(x) = \text{dist}(x, \partial\Omega_n)$ and C is a constant depending only on g and the C^2 character of Ω_n . Since $\{\Omega_n\}$ is uniformly C^2 , the constant may be chosen to be independent of n . Moreover $\{u_n\}$ is increasing. Therefore $u = \lim u_n \in L^1(\Omega) \cap L_\rho^g(\Omega)$ is the solution of (1.1). \square

Lemma 3.3. (a) *If $\mu \in \mathfrak{M}_\rho$ and $|\mu|$ is g -good then μ is g -good.* (b) *$T \in W^{-2} L^g(\Omega)$ if and only if $T = \Delta h$ for some $h \in L^g(\Omega)$.* (c) *If μ is a positive measure in $W^{-2} L_{\text{loc}}^g(\Omega)$ then $\mathbb{G}_\mu \in L_{\text{loc}}^g(\Omega)$. If, in addition, $\mu \in \mathfrak{M}_\rho(\Omega)$ then μ is g -good.*

Proof. (a) Assuming that $|\mu|$ is g -good, let v be the solution of (1.1) with μ replaced by $|\mu|$. Then v is a supersolution and $-v$ is a subsolution of (1.1). Therefore (1.1) has a solution.

(b) If $T = \Delta h$ then, for every $\phi \in C_c^\infty(\Omega)$,

$$T(\phi) = \int_{\Omega} h \Delta \phi dx, \quad |T(\phi)| \leq \|h\|_{L^g} \|\phi\|_{W^2 L^{g*}}.$$

As C_c^∞ is dense in $W_0^2 L^{g*}$, T defines a continuous linear functional on this space; consequently $T \in W^{-2} L^g(\Omega)$.

On the other hand if $T \in W^{-2}L^g(\Omega)$, put

$$S(\Delta\phi) := T(\phi) \quad \forall \phi \in W_0^2L^{g*}.$$

Note that for ϕ in this space we have $\phi = \mathbb{G}_{-\Delta\phi}$. Therefore S is well defined on the subspace of L^{g*} given by $\{\Delta\phi : \phi \in W_0^2L^{g*}\}$. Therefore there exists $h \in L^g(\Omega)$ such that

$$T(\phi) = \int_{\Omega} h \Delta\phi \, dx \quad \forall \phi \in W_0^2L^{g*}.$$

It follows that $T = \Delta h$.

(c) Let μ be a positive measure in $W^{-2}L_{\text{loc}}^g(\Omega)$. By part (b), if $\Omega' \Subset \Omega$ is a subdomain of class C^2 there exists $h \in L^g(\Omega')$ such that $\mu = \Delta h$. Then $h + \mathbb{G}_{\mu}$ is an harmonic function in Ω' ; consequently $\mathbb{G}_{\mu} \in L_{\text{loc}}^g(\Omega')$ and finally $\mathbb{G}_{\mu} \in L_{\text{loc}}^g(\Omega)$. If, in addition, $\mu \in \mathfrak{M}_{\rho}(\Omega)$ then, by Lemma 3.2, μ is g good. \square

Lemma 3.4. *Assume that $\mu \in \mathfrak{M}_{\rho}(\Omega)$ is g good. Then:*

- (i) *There exists $f \in L_{\rho}^1(\Omega)$ and $\mu_0 \in W^{-2}L_{\text{loc}}^g(\Omega) \cap \mathfrak{M}_{\rho}(\Omega)$ such that $\mu = f + \mu_0$.*
- (ii) *$\mu \prec C_{2,g^*}$.*

Proof. Assume that μ is g -good and let u be the solution of (1.1). Then

$$\mu = f + \mu_0 \quad \text{where } f := g \circ u \in L_{\rho}^1, \quad \mu_0 := \mu - g \circ u$$

and $u = \mathbb{G}_{\mu_0} \in L_{\rho}^g(\Omega)$. This implies that

$$\phi \mapsto \int_{\Omega} \phi \, d\mu_0 = \int_{\Omega} \Delta\phi u \, dx \quad \forall \phi \in C_c^{\infty}(\Omega)$$

is continuous on $C_0^2(\bar{\Omega})$ with respect to the norm of $W^2L_{\rho}^{g*}(\Omega)$. Therefore, the functional can be extended to a continuous linear functional on $W^2L^{g*}(\Omega')$ for every $\Omega' \Subset \Omega$. Thus $\mu_0 \in W^{-2}L_{\text{loc}}^g(\Omega) \cap \mathfrak{M}_{\rho}(\Omega)$. \square

(ii) In view of (2.3) it is sufficient to prove that μ vanishes on compact sets E such that $C_{2,g^*}(E) = 0$.

Assertion. *If $\nu \in W^{-2}L_{\text{loc}}^g(\Omega)$ then $\nu(E) = 0$ for every compact set E such that $C_{2,g^*}(E) = 0$.*

This assertion and part (i) imply part (ii).

Suppose that there exists a set E such that $C_{2,g^*}(E) = 0$ and $\nu(E) \neq 0$. Then there exists a compact subset of E on which ν has constant sign. Therefore we may assume that E is compact and that ν is positive on E . We may assume that $\nu \in W^{-2}L^g(\Omega)$; otherwise we replace Ω by a C^2 domain $\Omega' \Subset \Omega$.

Let $\{V_n\}$ be a sequence of open neighborhoods of E such that $\bar{V}_{n+1} \subset V_n$ and $V_n \downarrow E$. Then there exists a sequence $\{\varphi_n\}$ in $C_c^{\infty}(\Omega)$ such that $0 \leq \varphi_n \leq 1$, $\varphi_n = 1$ in V_{n+1} , $\text{supp } \varphi_n \subset V_n$ and $\|\varphi_n\|_{g^*} \rightarrow 0$.

This is proved in the same way as in the case of Bessel capacities. We use (2.5) and the fact that $C^{\infty}(\bar{\Omega})$ is dense in $W^2L_{\rho}^g(\Omega)$ [7]). Furthermore we

use an extension of the lemma on smooth truncation [1, Theorem 3.3.3] to Sobolev-Orlicz spaces with an integral number of derivatives. The extension is straightforward.

Hence,

$$(3.2) \quad \int_{\Omega} \varphi_n d\nu \rightarrow 0.$$

On the other hand,

$$\int_{\Omega} \varphi_n d\nu \geq \nu(\bar{V}_{n+1}) - |\nu|(V_n \setminus \bar{V}_{n+1}) \rightarrow \nu(E) > 0.$$

This contradiction proves the assertion. \square

Lemma 3.5. *Let μ be a positive measure in $\mathfrak{M}_\rho(\Omega)$. If μ vanishes on every compact set $E \subset \Omega$ such that $C_{2,g^*}(E) = 0$ then μ is the limit of an increasing sequence of positive measures $\{\mu_n\} \subset W^{-2}L^g(\Omega)$.*

Proof. Since μ is the limit of an increasing sequence of measures in $\mathfrak{M}(\Omega)$ it is sufficient to prove the lemma for $\mu \in \mathfrak{M}(\Omega)$. Let $\varphi \in W_0^2 L^{g^*}(\Omega)$ and denote

$$\tilde{\varphi} = \mathbb{G}_{\Delta\varphi}.$$

Then $\tilde{\varphi}$ is equivalent to φ .

Suppose that $\{\varphi_n\}$ converges to φ in $W_0^2 L^{g^*}(\Omega)$. Then $\Delta\varphi_n \rightarrow \Delta\varphi$ in L^{g^*} . Consequently, by [2, Theorem 4], there exists a subsequence such that $\tilde{\varphi}_{n'} \rightarrow \tilde{\varphi}$ C_{2,g^*} -a.e. (i.e., everywhere with the possible exception of a set of C_{2,g^*} -capacity zero). As μ vanishes on sets of capacity zero, it follows that $\tilde{\varphi}_{n'} \rightarrow \tilde{\varphi}$ μ -a.e..

Every $\varphi \in W_0^2 L^{g^*}(\Omega)$ is the limit of a sequence $\{\varphi_n\} \subset C_c^\infty(\Omega)$. Hence $\varphi_n \rightarrow \tilde{\varphi}$ μ -a.e. and consequently $\tilde{\varphi}$ is μ -measurable.

Therefore the functional $p : W_0^2 L^{g^*}(\Omega) \mapsto [0, \infty]$ given by

$$p(\varphi) := \int_{\Omega} (\tilde{\varphi})_+ d\mu$$

is well defined. The functional is sublinear, convex and l.s.c.: if $\varphi_n \rightarrow \varphi$ in $W_0^2 L^{g^*}(\Omega)$ then (by Fatou's lemma)

$$p(\varphi) \leq \liminf p(\varphi_n).$$

Furthermore,

$$p(a\varphi) = ap(\varphi) \quad \forall a > 0.$$

Therefore the result follows by an application of the Hahn-Banach theorem, in the same way as in [3, Lemma 4.2]. \square

Proof of Theorem 2.1. By Lemma 3.4 the condition $\mu \prec C_{2,g^*}$ is necessary for the existence of a solution. We show that the condition is sufficient.

If $\mu \prec C_{2,g^*}$ then $|\mu| \prec C_{2,g^*}$. By Lemma 3.3 if $|\mu|$ is g -good then μ is g -good. Therefore it remains to prove the sufficiency of the condition for positive μ . In this case, by Lemma 3.5, there exists an increasing sequence

of positive measures $\{\mu_n\} \subset W^{-2}L^g(\Omega)$ such that $\mu_n \uparrow \mu$. By Lemma 3.3 the measures μ_n are g -good. Denote by u_n the solution of (1.1) with μ replaced by μ_n . By Lemma 3.1, $u_n \geq 0$, $\{u_n\}$ increases and $\{u_n\}$ is bounded in $L^1(\Omega) \cap L_\rho^g(\Omega)$. Therefore $u = \lim u_n \in L^1(\Omega) \cap L_\rho^g(\Omega)$ and $u_n \rightarrow u$ in this space. Consequently u is the solution of (1.1). \square

4. PROOF OF THEOREM 2.3

(i) Let $\{O_n\}$ be a decreasing sequence of open sets such that $\bar{O}_{n+1} \subset O_n$, $\bar{O}_n \subset \Omega$ and $O_n \downarrow F$ and O_n is of class C^2 . By Theorem 2.1, the condition $\mu \prec C_{2,g^*}$ in $\Omega \setminus F$ is necessary and sufficient for the existence of a solution of the equation

$$(4.1) \quad -\Delta u + g \circ u = \mu \quad \text{in } \Omega_n := \Omega \setminus \bar{O}_n$$

such that $u = 0$ on the boundary. By a standard argument, it follows that, under this condition: for every $f \in L^1(\partial\Omega \cup \partial O_n)$, (4.1) has a solution such that $u = f$ on the boundary. As g satisfies the Keller – Osserman condition, it also follows that (4.1) has a solution u_n such that $u_n = 0$ on $\partial\Omega$ and $u_n = \infty$ on ∂O_n . Denote by v_n the solution of (4.1) vanishing on $\partial\Omega \cup \partial O_n$ and put

$$v_{0,\mu} = \lim v_n, \quad \bar{u}_\mu = \lim u_n.$$

Then $v_{0,\mu}$ is the smallest positive solution of (4.1) vanishing on $\partial\Omega$ while \bar{u}_μ is the largest such solution. In particular $\bar{u}_\mu \geq v_\nu$ for every $\nu \in \mathfrak{M}_\rho^g$ such that $\text{supp } \nu \subset F$. Thus \bar{u}_μ is the largest generalized solution of (1.1).

Next we construct the minimal generalized solution of (1.1). The function $u_{0,\mu} + V_F$ is a supersolution and $\max(u_{0,\mu}, V_F)$ is a subsolution of (4.1), both vanishing on the boundary. Let w_n denote the solution of (4.1) such that $w_n = 0$ on $\partial\Omega$ and $w_n = \max(u_{0,\mu}, V_F)$ on ∂O_n . Then

$$w_{n+1} \leq w_n \leq u_{0,\mu} + V_F$$

and consequently, $w = \lim w_n$ is the smallest solution of (4.1) such that

$$\max(u_{0,\mu}, V_F) \leq w \leq u_{0,\mu} + V_F.$$

It follows that w is a generalized solution of (1.1). Since any such solution dominates $\max(u_{0,\mu}, V_F)$ it follows that w is the smallest generalized solution of the problem. It is easy to see that $w = \underline{u}_\mu$ as given by (2.7).

Since g is convex, monotone increasing and $g(0) = 0$ we have

$$g(a) + g(b) \leq g(a + b) \quad \forall a, b \in \mathbb{R}_+.$$

Therefore $\bar{u}_\mu - u_{0,\mu}$ is a subsolution of (2.6) in $\Omega \setminus F$. Consequently $\bar{u}_\mu - u_{0,\mu} \leq U_F$ and

$$(4.2) \quad \max(u_{0,\mu}, U_F) \leq \bar{u}_\mu \leq u_{0,\mu} + U_F.$$

Put $\Omega_n = \Omega \setminus \bar{O}_n$. Let \underline{u}_n be the solution of the problem

$$\begin{aligned} -\Delta u + g \circ u &= \mu \quad \text{in } \Omega_n, \\ u &= V_F \quad \text{on } \partial O_n, \quad u = 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Then $\{\underline{u}_n\}$ increases and $\underline{u} = \lim \underline{u}_n$.

Similarly, if \bar{u}_n is the solution of the problem

$$\begin{aligned} -\Delta u + g \circ u &= \mu \quad \text{in } \Omega_n, \\ u &= U_F \quad \text{on } \partial\Omega_n, \quad u = 0 \quad \text{on } \partial\Omega. \end{aligned}$$

then $\{\bar{u}_n\}$ increases and, in view of (4.2), $\bar{u} = \lim \bar{u}_n$. Therefore, if $V_F = U_F$ then $\bar{u}_\mu = \underline{u}_\mu$.

(ii) We assume that in addition to the other conditions of the theorem, g satisfies the subcriticality condition. In this case, for every point $z \in \Omega$ and $k \in \mathbb{R}$, there exists a solution $u_{k,z}$ of the problem

$$(4.3) \quad -\Delta u + g \circ u = k\delta_z \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Put $w_z = \lim_{k \rightarrow \infty} u_{k,z}$. By definition $w_z = V_{\{z\}}$. We also have $w_z = U_{\{z\}}$. This follows from the fact that g satisfies the Keller – Osserman condition. This condition implies that there exists a decreasing function $\psi \in C(0, \infty)$ such that $\psi(t) \rightarrow \infty$ as $t \rightarrow 0$ and every positive solution u of (4.3) satisfies

$$C_2\psi(|x - z|) \leq u(x) \leq C_1\psi(|x - z|)$$

The constant C_1 depends only on g, N . Because of the boundary condition the constant C_2 depends on z . However for z in a compact subset of Ω one can choose C_2 to be independent of z .

This inequality implies that

$$w_z \leq U_{\{z\}} \leq C_1/C_2 w_z.$$

If F is a compact subset of Ω put

$$F' = \{x \in \Omega : \text{dist}(x, F) \leq \frac{1}{2} \text{dist}(F, \partial\Omega)\}.$$

Let $x \in F' \setminus F$ and let z be a point in F such that $|x - z| = \text{dist}(x, F)$. Then there exists a positive constant $C(F)$ such that

$$C(F)\psi(|x - z|) \leq U_z(x) \leq V_F(x) \leq U_F(x) \leq C_1\psi(|x - z|).$$

It follows that there exists a constant c such that

$$(4.4) \quad U_F(x) \leq cV_F(x)$$

for every $x \in F'$. Since U_F and V_F vanish on $\partial\Omega$ it follows that (4.4) (with possibly a larger constant) remains valid in $\Omega \setminus F'$. This is verified by a standard argument using Harnack's inequality and the fact that g satisfies the Keller – Osserman condition. Thus (4.4) is valid in $\Omega \setminus F$. By an argument similar to the one introduced in [10, Theorem 5.4], this inequality implies that $U_F = V_F$.

(iii) For the case considered here, it was proved in [11] that $U_F = V_F$. Therefore uniqueness follows from part (i). \square

REFERENCES

- [1] Adams D. R. and Hedberg L. I., Function spaces and potential theory, Grundlehren Math. Wissen. **314**, Springer (1996).
- [2] Aissaoui N. and Benkirane A., *Capacité dans les espaces d'Orlicz*, Ann. Sci. Math. Québec 12 (1994) 1-23.
- [3] Baras P., Pierre M. *Singularités éliminables pour des équations semilinéaires*, Ann. Inst. Fourier 34 (1984), 185-206.
- [4] Benilan Ph. and Brezis H., *Nonlinear problems related to the Thomas-Fermi equation*, J. Evolution Eq. 3, 673-770 (2003).
- [5] Brezis H., Notes (circa 1970) unpublished.
- [6] Brezis H. and Strauss W., *Semilinear second-order elliptic equations in L^1* , J. Math. Soc. Japan 25 (1973), 565-590.
- [7] Donaldson Th. K. and Trudinger N. S. *Orlicz-Sobolev spaces and imbedding theorems*, J. Funct. An. 8 (1971) 52-75.
- [8] Krasnoselskii M. A. and Rutickii Y. B., Convex functions and Orlicz spaces, P. Noordhoff, Groningen, 1961.
- [9] Keller J. B. , *On solutions of $\Delta u = f(u)$* , Comm. Pure Appl. Math. 10, 503-510 (1957).
- [10] Marcus M. and Véron L., *The boundary trace of positive solutions of semilinear elliptic equations: the subcritical case*, Arch. rat. Mech. Anal. **144**, 201-231 (1998).
- [11] Marcus M. and Véron L., *Capacitary estimates of positive solutions of semilinear elliptic equations with absorption*, J. European Math. Soc. **6**, 483-527 (2004).
- [12] Osserman R., *On the inequality $\Delta u \geq f(u)$* , Pacific J. Math. 7, 1641-1647 (1957).
- [13] Véron L., Singularities of solutions of second order quasilinear equations, Pitman Research Notes, vol. 353, Longman, 1996.

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